

ON ALZER'S INEQUALITY

MOHAMMAD W. ALOMARI

ABSTRACT. Extensions and generalizations of Alzer's inequality; which is of Wirtinger type are proved. As applications, sharp trapezoid type inequality and sharp bound for the geometric mean are deduced.

1. INTRODUCTION

In Fourier analysis, the theory of inequalities plays an important and useful role in almost all branches of its analyzes. Early of the last century, several famous inequalities have been used in the theory of Fourier series, Fourier integrals and Fourier transform. The inequalities of Bessel, Blaschke, Wirtinger, Beesack and others, are used at large in convergence and estimations of such series and integrals.

In [4], Wirtinger proved the following inequality regarding square integrable functions:

Theorem 1. *Let f be a real valued function with period 2π and $\int_0^{2\pi} f(x) dx = 0$. If $f' \in L^2[0, 2\pi]$, then*

$$(1.1) \quad \int_0^{2\pi} f^2(x) dx \leq \int_0^{2\pi} f'^2(x) dx,$$

with equality if and only if $f(x) = A \cos x + B \sin x$, $A, B \in \mathbb{R}$.

Various generalizations, counterparts and refinements were considered in [1]–[6] and the references therein.

In [1], Alzer introduced a Wirtinger like inequality for continuously differentiable periodic functions, which reads:

Theorem 2. *If f is a real valued continuously differentiable function with period 2π and $\int_0^{2\pi} f(x) dx = 0$, then*

$$(1.2) \quad \frac{6}{\pi} \max_{0 \leq x \leq 2\pi} f^2(x) \leq \int_0^{2\pi} f'^2(x) dx.$$

Equality holds if and only if $f(x) = c \left[3 \left(\frac{x-\pi}{\pi} \right)^2 - 1 \right]$, $0 \leq x \leq 2\pi$ and $c \in \mathbb{R}$.

The aim of this work is to extend and generalize Alzer inequality (1.2), by relaxing the assumptions: continuity of f' , periodicity and the interval involved for various kind of functions.

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2. THE RESULTS

The version of Alzer inequality for convex functions may be stated as follows:

Theorem 3. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f' \in L[a, b]$. If $f(a)f(b) > 0$ and $\int_a^b f(t) dt = 0$, then the inequality*

$$(2.1) \quad f(a)f(b) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

holds. The constant $\frac{b-a}{12}$ is the best possible, in the sense that it cannot be replaced by a smaller constant.

Proof. Assume that f attains its maximum value at $x_0 \in [a, b]$ and let $\max_{a \leq x \leq b} f(x) = f(x_0)$, for some $a \leq x_0 \leq b$, then

$$(2.2) \quad \begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{24}{(b-a)^2 f(x_0)} \int_a^b \left(x - \frac{a+b}{2} \right) f'(x) dx \\ &\quad + \frac{144}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2} \right)^2 dx. \end{aligned}$$

Observing that

$$\int_a^b \left(x - \frac{a+b}{2} \right) f'(x) dx = \frac{b-a}{2} \cdot [f(a) + f(b)] - \int_a^b f(x) dx,$$

taking in account that $\int_a^b f(x) dx = 0$. Substituting in (2.2), we get

$$\begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{12}{(b-a) f(x_0)} [f(a) + f(b)] + \frac{12}{(b-a)} \end{aligned}$$

which gives that

$$\{f(a) + f(b) - f(x_0)\} \cdot \max_{a \leq x \leq b} f(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx.$$

Finally, since f is convex then f attains its maximum at the endpoints 'a' or 'b', so if $\max_{a \leq x \leq b} f(x) = f(b) = f(x_0)$, thus we have

$$(2.3) \quad f(a) \cdot \max_{a \leq x \leq b} f(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

and if $\max_{a \leq x \leq b} f(x) = f(a) = f(x_0)$, we have

$$(2.4) \quad f(b) \cdot \max_{a \leq x \leq b} f(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx.$$

So that the both inequalities (2.3) and (2.4), can be read as

$$f(a) f(b) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

and thus the proof of (2.11) is established. To prove the sharpness of (2.11), let (2.11) holds with another constant $C > 0$,

$$(2.5) \quad f(a) f(b) \leq C \cdot \int_a^b f'^2(x) dx.$$

Define the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = 6x^2 - 6x + 1$, for all $x \in [0, 1]$. Clearly, f is convex for all $x \in [0, 1]$. Moreover, we have $f(0) = f(1) = 1$, and $\int_0^1 f'^2(x) dx = 12$. Making use of (2.5), we have $C \geq \frac{1}{12}$, and this proves the best possibility of $\frac{1}{12}$, which completes the proof. \square

The following inequality for monotonic mappings holds.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f' \in L[a, b]$. If $\int_a^b f(t) dt = 0$, then the inequality*

$$(2.6) \quad [2f(a) - f(b)] \cdot f(b) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

holds. The constant $\frac{b-a}{12}$ is the best possible.

Proof. Repeating the steps in the proof of Theorem 7, since f is bounded and monotonically increasing on $[a, b]$, then $f(a) \leq f(t)$ for all $t \in [a, b]$, therefore

$$\begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{12}{(b-a)f(x_0)} [f(a) + f(b)] + \frac{12}{(b-a)} \\ &\leq \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{24}{(b-a)f(x_0)} f(a) + \frac{12}{(b-a)} \end{aligned}$$

which gives that

$$[2f(a) - f(b)] \cdot f(b) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

which proves the inequality (2.6). The sharpness holds with the function $f(x) = 4c^2 \cdot x^3 + 12c \cdot x - c^2 - 6c$, for all $x \in [0, 1]$, where $c = \frac{10+2\sqrt{835}}{27}$. \square

Corollary 1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a bounded decreasing function on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f' \in L[a, b]$. If $\int_a^b f(t) dt = 0$, then the inequality*

$$(2.7) \quad [2f(b) - f(a)] \cdot f(a) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

holds. The constant $\frac{b-a}{12}$ is the best possible.

Proof. The proof is similar the proof of Theorem 4. \square

In general, we may generalize and extend Alzer inequality (1.2) as follows:

Theorem 5. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f' \in L[a, b]$. If $f(a) = \max_{a \leq x \leq b} f(x) = f(b)$ and $\int_a^b f(t) dt = 0$, then the inequality*

$$(2.8) \quad \max_{a \leq x \leq b} f^2(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

holds. The constant $\frac{b-a}{12}$ is the best possible.

Proof. Given the assumptions. Assume that f attains its maximum value at $x_0 \in [a, b]$ and let $\max_{a \leq x \leq b} f(x) = f(x_0)$, for some $a \leq x_0 \leq b$, then

$$(2.9) \quad \begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{24}{(b-a)^2 f(x_0)} \int_a^b \left(x - \frac{a+b}{2} \right) f'(x) dx \\ &\quad + \frac{144}{(b-a)^4} \int_a^b \left(x - \frac{a+b}{2} \right)^2 dx. \end{aligned}$$

Since $f(a) = \max_{a \leq x \leq b} f(x) = f(b)$, we have

$$\int_a^b \left(x - \frac{a+b}{2} \right) f'(x) dx = \frac{b-a}{2} \cdot [f(a) + f(b)] = (b-a) \cdot f(x_0)$$

Substituting in (2.9),

$$\begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx = \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{24}{b-a} + \frac{12}{b-a} \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{12}{b-a} \end{aligned}$$

which gives that

$$\max_{a \leq x \leq b} f^2(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

and thus the proof of (2.9) is established. To prove the sharpness of (2.8), let $a = 0$, $b = 2\pi$, then (2.8) reduces to (1.2), so by considering the same function f as given in Theorem 2, we get the sharpness. \square

The most extensive case holds without any additional restrictions on f is considered as follows:

Theorem 6. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f' \in L[a, b]$. Then the inequality*

$$(2.10) \quad \left[\frac{2}{b-a} \cdot \mathcal{T}_{\text{rap}}(f) - \max_{a \leq x \leq b} f(x) \right] \cdot \max_{a \leq x \leq b} f(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

holds, where

$$\mathcal{T}_{\text{rap}}(f) := (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(x) dx.$$

The inequality is sharp.

Proof. Repeating the steps in the proof of Theorem 5 taking in account that no restrictions on f , we have

$$\begin{aligned} 0 &\leq \int_a^b \left[\frac{f'(x)}{f(x_0)} - \frac{12}{(b-a)^2} \cdot \left(x - \frac{a+b}{2} \right) \right]^2 dx \\ &= \int_a^b \frac{f'^2(x)}{f^2(x_0)} dx - \frac{24}{(b-a)^2 f(x_0)} \cdot \mathcal{T}_{\text{rap}}(f) + \frac{12}{(b-a)} \end{aligned}$$

which gives that

$$\frac{2\mathcal{T}_{\text{rap}}(f) - (b-a)f(x_0)}{b-a} \cdot \max_{a \leq x \leq b} f(x) \leq \frac{b-a}{12} \cdot \int_a^b f'^2(x) dx,$$

and thus the proof of (2.10) is established. The sharpness follows with $f(x) = 6x^2 - 6x + 1$, for all $x \in [0, 1]$. \square

Another generalization for $(2n)$ -times differentiable functions is considered as follows:

Theorem 7. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be $(2n)$ -times differentiable ($n \geq 1$) on I° , the interior of the interval I , where $a, b \in I^\circ$ with $a < b$, such that $f^{(2n)} \in L^1[a, b]$. If $\int_a^b f(t) dt = 0$, then the inequality

$$(2.11) \quad \|f\|_\infty \leq \left(\frac{b-a}{12} \right)^n \cdot \|f^{(2n)}\|_2$$

holds, where, $\|f\|_\infty := \sup_{a \leq x \leq b} |f(x)|$ and $\|f^{(2n)}\|_2^2 = \int_a^b |f^{(2n)}(x)|^2 dx$.

Proof. Setting

$$\alpha = \frac{(12)^n (b-a)^{-(n+\frac{1}{2})}}{B^{\frac{1}{2}}(2n+1, 2n+1)} \quad n \in \mathbb{N},$$

where $B(\cdot, \cdot)$ is Euler-beta function. Assume that f attains its maximum value at $x_0 \in [a, b]$ and let $\sup_{a \leq x \leq b} f(x) = f(x_0)$, for some $a \leq x_0 \leq b$, then

$$\begin{aligned} 0 &\leq \int_a^b \left[\frac{f^{(2n)}(x)}{f(x_0)} - \alpha \cdot \frac{(x-a)^n (b-x)^n}{(b-a)^{2n}} \right]^2 dx \\ &= \int_a^b \frac{(f^{(2n)}(x))^2}{f^2(x_0)} dx - \frac{2\alpha}{(b-a)^{2n} f(x_0)} \int_a^b (x-a)^n (b-x)^n f^{(2n)}(x) dx \\ &\quad + \frac{\alpha^2}{(b-a)^{4n}} \int_a^b (x-a)^{2n} (b-x)^{2n} dx \end{aligned}$$

Therefore,

$$\begin{aligned}
 \int_a^b \left(f^{(2n)}(x) \right)^2 dx &\geq \frac{2\alpha}{(b-a)^{2n}} f(x_0) \int_a^b (x-a)^n (b-x)^n f^{(2n)}(x) dx \\
 &\quad - \frac{\alpha^2}{(b-a)^{4n}} f^2(x_0) \int_a^b (x-a)^{2n} (b-x)^{2n} dx \\
 (2.12) \qquad &= \frac{2\alpha}{(b-a)^{2n}} f(x_0) \int_a^b (x-a)^n (b-x)^n f^{(2n)}(x) dx \\
 &\quad - \alpha^2 (b-a) f^2(x_0) B(2n+1, 2n+1).
 \end{aligned}$$

It is not difficult to observe that

$$\int_a^b (x-a)^n (b-x)^n f^{(2n)}(x) dx = 0,$$

which follows by integrating by parts and using the given assumptions.

Now, by triangle inequality we have

$$\begin{aligned}
 \int_a^b \left| f^{(2n)}(x) \right|^2 dx &\geq \left| \int_a^b \left(f^{(2n)}(x) \right)^2 dx \right| \\
 &\geq \alpha^2 (b-a) |f(x_0)|^2 B(2n+1, 2n+1),
 \end{aligned}$$

simple computations gives the required result (2.11). \square

3. USEFUL APPLICATIONS

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$, be a twice differentiable mapping such that $f''(x)$ exists on I° , and $\|f''\|_\infty = \sup_{x \in (a,b)} |f''(x)| < \infty$. Then the trapezoid inequality

$$(3.1) \qquad \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(x) dx \right| \leq \frac{(b-a)^3}{12} \|f''\|_\infty,$$

holds. Therefore, the integral $\int_a^b f(x) dx$ can be approximated in terms of the trapezoidal rules, respectively such as:

$$\int_a^b f(x) dx \cong (b-a) \frac{f(a) + f(b)}{2}.$$

By means of (2.10), it is significant to remark that the inequality has a trapezoid bound term, therefore we may rewrite (2.10) to obtain a new upper bound for the trapezoid inequality, such as:

Corollary 2. *Under the assumptions of Theorem 6, we have*

$$(3.2) \qquad \mathcal{T}_{\text{rap}}(f) \leq \frac{b-a}{2} \cdot \max_{a \leq x \leq b} f(x) + \frac{(b-a)^2}{24 \cdot \max_{a \leq x \leq b} f(x)} \cdot \int_a^b f'^2(x) dx,$$

provided that $\max_{a \leq x \leq b} f(x) \neq 0$. Equivalently, in terms of norms we may write

$$(3.3) \qquad |\mathcal{T}_{\text{rap}}(f)| \leq \frac{b-a}{2} \cdot \|f\|_\infty + \frac{(b-a)^2}{24} \cdot \frac{\|f'\|_2^2}{\|f\|_\infty},$$

where; $\|f\|_\infty = \sup_{a \leq x \leq b} |f(x)|$ and $\|f'\|_2^2 = \int_a^b |f'(x)|^2 dx$. The two inequalities are sharp.

Henceforth, by setting

$$M := \frac{6}{(b-a)^2} \cdot \|f\|_\infty + \frac{1}{2(b-a)} \cdot \frac{\|f'\|_2^2}{\|f\|_\infty},$$

a beautiful trapezoid inequality may be written as:

$$(3.4) \quad \left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(x) dx \right| \leq \frac{(b-a)^3}{12} M,$$

which holds with more less restrictions on f , and so if f is twice differentiable and has bounded second derivative, with $M \leq \|f''\|_\infty$, then totally (3.4) by its assumptions can be better than (3.1), and exactly if $M := \|f''\|_\infty$. So that we have applied our result (2.10) to obtain new trapezoid type inequality which has important applications in numerical integrations.

One more direct interesting application is to bound the geometric mean by a sharp upper bound. This happens if one assumes $f(a)f(b) > 0$, which already holds by assumptions of Theorem 7, then (2.11) can be written as:

$$\sqrt{f(a)f(b)} \leq \sqrt{\frac{b-a}{12}} \cdot \left(\int_a^b f'^2(x) dx \right)^{1/2},$$

equivalently we write,

$$(3.5) \quad G(f(a), f(b)) \leq \sqrt{\frac{b-a}{12}} \cdot \|f'\|_2,$$

where $G(\cdot, \cdot)$ is the geometric mean and the inequality is sharp.

Moreover, if f is log-convex, i.e., f satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq f^\lambda(x) f^{1-\lambda}(y).$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. In particular, choose $\lambda = \frac{1}{2}$, then the double inequality

$$(3.6) \quad f(A(x, y)) \leq G(f(x), f(y)) \leq \sqrt{\frac{y-x}{12}} \cdot \|f'\|_2,$$

holds and sharp; provided that $a \leq x < y \leq b$, where $A(\cdot, \cdot)$ is the arithmetic mean. Clearly, the left-hand side inequality sharp by the definition of log-convexity.

A generalization of this result can be done if f is considered to be bijective on $[a, b]$. Choosing $\alpha, \beta \in [a, b]$ such that $f(\alpha) = f^\lambda(x)$ and $f(\beta) = f^{1-\lambda}(y)$, for some $\lambda \in [0, 1]$ and $x, y \in [a, b]$. Making use of (2.11) we have

$$(3.7) \quad f(\alpha) f(\beta) \leq \frac{\beta - \alpha}{12} \int_\alpha^\beta f'^2(x) dx.$$

Therefore, a generalization of (3.6) may given as:

$$(3.8) \quad \begin{aligned} f(\lambda x + (1-\lambda)y) &\leq f^\lambda(x) f^{1-\lambda}(y) \\ &\leq \frac{f^{-1}(f^{1-\lambda}(y)) - f^{-1}(f^\lambda(x))}{12} \cdot \int_{f^\lambda(x)}^{f^{1-\lambda}(y)} f'^2(x) dx. \end{aligned}$$

or written in terms of generalized means, as

$$f(A_\lambda(x, y)) \leq G_\lambda(f(x), f(y)) \leq \frac{f^{-1}(f^{1-\lambda}(y)) - f^{-1}(f^\lambda(x))}{12} \cdot \int_{f^\lambda(x)}^{f^{1-\lambda}(y)} f'^2(x) dx$$

where, $A_\lambda(x, y) = \lambda x + (1 - \lambda)y$, is the generalized arithmetic mean and $G_\lambda(x, y) = x^\lambda y^{1-\lambda}$, is the generalized geometric mean.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND INFORMATION TECHNOLOGY, IRBID
NATIONAL UNIVERSITY, P.O. BOX 2600, IRBID, P.C. 21110, JORDAN.
E-mail address: mwomath@gmail.com